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A calculus based on a q-deformed Heisenberg algebra

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Abstract. We show how one can construct a differential calculus over an algebra where position variables x and momentum variables p have be defined. As the simplest example we consider the one-dimensional q-deformed Heisenberg algebra. This algebra has a subalgebra generated by x and its inverse which we call the coordinate algebra. A physical field is considered to be an element of the completion of this algebra. We can construct a derivative which leaves invariant the coordinate algebra and so takes physical fields into physical fields. A generalized Leibniz rule for this algebra can be found. Based on this derivative differential forms and an exterior differential calculus can be constructed.

1 Introduction

We would like to show how one can construct a differential calculus over an algebra where position variables x and momentum variables p have been defined. As the simplest example we consider the one-dimensional q-deformed Heisenberg algebra [1]. This algebra has a subalgebra generated by x and its inverse which we call the coordinate algebra. A physical field is defined to be an element of the completion of the algebra. We can construct a derivative which leaves invariant the elements of the coordinate algebra and so takes physical fields into physical fields. A generalized Leibniz rule for this algebra can be found. Using this derivative differential forms and an exterior differential calculus can be constructed. This is done in Sect. 4.

On the coordinate algebra it is possible to define an integral purely algebraically, as the inverse image of the derivative; to a given function we associate another function as the integral . We study the above definition of the integral in an explicit representation of the algebra, a large class of which are known [2]. This leads us in a natural way, in Sect. 5, to a trace formula for the integral which produces the well known Jackson integral [3]. We find that a form of Stokes' theorem can be proven. There is an interesting fact about the summation in the formula for the Jackson integral: even and odd lattice points must be treated separately. The integral separates therefore into a sum over even lattice points and a sum over odd lattice points. Such a phenomenon is not unusual for the q-deformed Heisenberg algebra [2]. The separation can be traced to the fact that the derivative is actually a second-order differential operator [4]. With the integral we define in Sect. 6 an inner product and therefore a Hilbert space L^2_q and so the notion of an hermitian and a self-adjoint operator becomes meaningful. Differentiation becomes a linear operator on L^2_q. It is not hermitian but its square has a self-adjoint extension. In Sect. 7 we introduce a basis for L^2_q in terms of q-cosine and q-sine functions [5]. This opens the way for quantum mechanics in the form of wave mechanics. In Sect. 8 a Schrödinger equation is defined. This allows a probability interpretation with a probability density and a probability density current which satisfy a continuity equation.

The derivatives which act on a wave function can be generalized to derivatives which are covariant under the action of a gauge transformation with parameters which depend on the lattice point. It turns out that the corresponding connection can be defined in terms of a "moving frame" which we call, in Sect. 9, an Einbein. In Sect. 10 we discuss the Leibniz rules for these covariant derivatives. In Sect. 11 we introduce also a covariant time derivative and construct a field strength from the commutator of the two. This permits us to give a differential geometric treatment of our lattice structure. In Sect. 12 it is shown that using our concept of integration it is possible to derive the Schrödinger equation from a Lagrangian and a variational principle as well as a Noether theorem.

Finally, in Sect. 13, we treat the q-deformed harmonic oscillator as a quantum mechanical example in this formalism. Formally this is similar to the usual treatment of the harmonic oscillator with creation and annihilation operators [6]. The wave functions can be found by solving q-differential equations. The Gauss function on the lattice plays an important role. We define a Fourier transformation with the q-cosine and q-sine functions. The ground state of the harmonic oscillator is a Gaussian in the momentum space. Under a Fourier transformation it is no longer a Gaussian; it is the q-exponential function. This provides a nice example where a q- Fourier transformation can be carried out explicitly.

2 The algebra and its representation

The q-deformed Heisenberg algebra on which our calculus will be based is a formal $*$-algebra generated by elements (x, p) and an extra generator Λ (the dilatator) which satisfy the commutation relations

$$q^{\frac{1}{2}}xp - q^{-\frac{1}{2}}px = i\Lambda, \tag{2.1}$$
$$\Lambda p = qp\Lambda, \quad \Lambda x = q^{-1}x\Lambda \tag{2.2}$$

Here q is a real number greater than one. The elements x and p are assumed to be hermitian and Λ to be unitary:

$$\bar{x} = x, \quad \bar{p} = p, \quad \bar{\Lambda} = \Lambda^{-1} \tag{2.3}$$

This "bar" operation is meant to be an algebraic involution and coincides with complex conjugation on numbers ($\bar{q} = q$). This algebra and its representations have been studied by Hebecker *et al.* [2] and Schmüdgen [4]. In these representations the bar operation is realized as the star operation on linear operators. We are interested only in those representations where a formally hermitian operator is represented by a self-adjoint linear operator on L^2_q. In such representations the operator x can be assumed to be diagonal and its eigenvalues are [2] given by

$$x|n,\sigma\rangle^s = \sigma s q^n |n,\sigma\rangle^s \tag{2.4}$$
$$n \in \mathbb{Z}, \quad \sigma = \pm 1$$

The number s characterizes the representation and can take the values $1 \leq s < q$. The eigenvectors $|n,\sigma\rangle^s$ form an orthonormal basis of L^2_q:

$${}^s\langle n',\sigma'|n,\sigma\rangle^s = \delta_{n,n'}\delta_{\sigma,\sigma'} \tag{2.5}$$

The action of the operator Λ in the above representation is given by

$$\Lambda|n,\sigma\rangle^s = |n+1,\sigma\rangle^s \tag{2.6}$$

The action of p can now be obtained from (2.4) and (2.6). We must first enlarge the algebra by adding the element x^{-1}. This element is well defined on the basis $|n,\sigma\rangle^s$. Next we conjugate the relation (2.1):

$$q^{\frac{1}{2}}px - q^{-\frac{1}{2}}xp = -i\Lambda^{-1} \tag{2.7}$$

When we eliminate px from the equations (2.1) and (2.7) we obtain

$$xp = i\frac{q^{\frac{1}{2}}}{(q-q^{-1})}(\Lambda - q^{-1}\Lambda^{-1}) \tag{2.8}$$

We multiply this relation by x^{-1} and we find that

$$p = i\frac{q^{\frac{1}{2}}}{(q-q^{-1})}x^{-1}(\Lambda - q^{-1}\Lambda^{-1}) \tag{2.9}$$

For the representations (2.4) and (2.6) this yields (s=1)

$$p|n,\sigma\rangle = i\frac{\sigma}{(q-q^{-1})}q^{-n}\left(q^{-\frac{1}{2}}|n+1,\sigma\rangle - q^{\frac{1}{2}}|n-1,\sigma\rangle\right) \tag{2.10}$$

We note that σ does not change under the action of x, p and Λ. Therefore each sign of σ yields a representation. But both signs of σ must be used to find self-adjoint extensions of the hermitian operator p, as it is defined by (2.10), which then also satisfy the relations (2.1).

The algebraic relations allow an arbitrary ordering of the elements $(x,\ p,\ \Lambda)$; any product of these elements, in arbitrary order, can be expressed in terms of ordered polynomials, for example in the order $xp\Lambda$. We notice also that the algebra defined by the relations (2.1) has a subalgebra which is generated by the elements $(p, p^{-1}, \Lambda, \Lambda^{-1})$ as well as a subalgebra generated by the elements (x, x^{-1}). Elements of this latter algebra we shall call fields. By $f(x)$ is meant an element of the algebra generated by (x, x^{-1}) which then is completed by allowing formal power series.

3 Derivatives

In the previous section we defined fields. Derivatives will be mappings of this algebra into itself which we shall now define. From the ordering property of the algebra we know that for any field f there are fields g and h such that

$$pf(x) = g(x)p - iq^{\frac{1}{2}}h(x)\Lambda \tag{3.1}$$

as well as a field $j(x)$ such that

$$\Lambda f(x) = j(x)\Lambda \tag{3.2}$$

A derivative is now defined as the map

$$\nabla f(x) = h(x) \tag{3.3}$$

In addition we define

$$Lf(x) = j(x) \tag{3.4}$$

On the monomials $x^n, (n \in \mathbb{Z})$ these maps are given by

$$\nabla x^n = [n]x^{n-1}, \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{3.5}$$
$$Lx^n = q^{-n}x^n$$

These monomials form a basis of the algebra generated by (x, x^{-1}). It follows therefore from (3.5) that there is an algebra morphism of the (p, Λ) algebra to the (∇, L) algebra define by the relation

$$L\nabla = q\nabla L \tag{3.6}$$

It follows also from (3.5) that the action of ∇ can be generated by L, L^{-1} and x^{-1}:

$$\nabla = \frac{1}{q-q^{-1}}x^{-1}(L^{-1} - L) \tag{3.7}$$

In this formula x^{-1} is to be interpreted as a map defined by left multiplication within the (x, x^{-1}) algebra.

Next we consider the Leibniz rule. From the formula

$$x^{m+n} = x^m x^n \tag{3.8}$$

it follows that

$$(Lx^{m+n}) = (Lx^m)(Lx^n) \qquad (3.9)$$
$$(L^{-1}x^{m+n}) = (L^{-1}x^m)(L^{-1}x^n)$$

From the expression (3.7) for ∇ we obtain from (3.9) the Leibniz rule

$$\begin{aligned}\nabla(fg) &= (\nabla f)(Lg) + (L^{-1}f)(\nabla g) \\ &= (\nabla f)(L^{-1}g) + (Lf)(\nabla g)\end{aligned} \qquad (3.10)$$

for the derivative. Equations (3.9) and (3.10) can be seen as comultiplication rules for the (∇, L) algebra:

$$\begin{aligned}\Delta(\nabla) &= \nabla \otimes L + L^{-1} \otimes \nabla \\ \Delta(L) &= L \otimes L\end{aligned} \qquad (3.11)$$

or

$$\begin{aligned}\Delta(\nabla) &= \nabla \otimes L^{-1} + L \otimes \nabla \\ \Delta(L) &= L \otimes L\end{aligned} \qquad (3.12)$$

It is easy to see that this is an algebra morphism. Acting on fields the two comultiplication rules of ∇ coincide. The map defined by (3.3) is not onto since x^{-1} is not in the image of ∇. The kernel of this map consists of the constants:

$$\nabla c = 0 \qquad (3.13)$$

4 Differentials

We define the differential or exterior derivative on elements of the (x, x^{-1}) algebra as

$$d = dx\,\nabla, \quad dx = (dx) \qquad (4.1)$$

It has a unique extension to 1-forms if one imposes $d^2 = 0$. Because x and dx can be ordered there are no higher-order forms. If we apply d to x^n we obtain

$$dx^n = dx\,\nabla x^n = [n] dx\, x^{n-1} \qquad (4.2)$$

To find a Leibniz rule we start from the relation:

$$dx^{m+n} = [m+n] dx\, x^{m+n-1} \qquad (4.3)$$

and we try the Ansatz

$$dx^m x^n = dx^m (Ax^n) + (Bx^m) dx^n \qquad (4.4)$$

This can be compared with (4.3):

$$[m] dx\, x^{m-1}(Ax^n) + (Bx^m)[n] dx\, x^{n-1} = [m+n] dx\, x^{m+n-1} \qquad (4.5)$$

We collect the terms with dx on the left hand side:

$$dx\Big([m+n]x^{m+n-1} - [m]x^{m-1}(Ax^n)\Big) = [n](Bx^m) dx\, x^{n-1} \qquad (4.6)$$

We find that the left hand side has to be proportional to $[n]$. This can be achieved by setting $A = L^{-1}$ or $A = L$. We first analyze the case $A = L^{-1}$

$$(Ax^n) = (L^{-1}x^n) = q^n x^n \qquad (4.7)$$

Eqn (4.6) becomes

$$dx\, q^{-m} x^{m+n-1} = (Bx)^m dx\, x^{n-1} \qquad (4.8)$$

or

$$dx\, x^m = q^m (Bx)^m dx \qquad (4.9)$$

If we set $B = L^b$ with $b \in \mathbb{Z}$ we can derive from (4.9) the commutation relation

$$dx\, x = q^{1-b} x dx \qquad (4.10)$$

between dx and x. This leads us to the Leibniz rule

$$\begin{aligned}d(fg) &= df(L^{-1}g) + (L^b f)dg \\ dx\, x &= q^{1-b} x dx\end{aligned} \qquad (4.11)$$

The second choice for A is $A = L$. The equation analogous to (4.8) is now

$$dx\, q^m x^{m+n-1} = (Bx)^m dx\, x^{n-1} \qquad (4.12)$$

or

$$dx\, x^m = q^{-m}(Bx)^m dx. \qquad (4.13)$$

We obtain the Leibniz rule:

$$\begin{aligned}d(fg) &= df(Lg) + (L^b f)dg \\ dx\, x &= q^{-1-b} x dx\end{aligned} \qquad (4.14)$$

An interesting choice for the Leibniz rule (4.11) is $b = 1$ or for (4.14) $b = -1$; in both cases dx and x commute.

5 The integral

We define the indefinite integral to be the inverse image of (3.3). Integrals have also been defined by A. Kempf and S. Majid in [7] and by H. Steinacker in [8]. The kernel of the map (3.3) are the constants and x^{-1} is not in the image of ∇. Thus we find

$$\int^x x^n = \frac{1}{[n+1]} x^{n+1} + \text{const.} \qquad n \in \mathbb{Z},\, n \neq -1 \quad (5.1)$$

A useful formula is obtained if we invert ∇ in the form (3.7):

$$\nabla^{-1} = (q - q^{-1}) \frac{1}{L^{-1} - L}\, x \qquad (5.2)$$

For $m \neq -1$ we can apply this to x^m and obtain

$$\begin{aligned}\nabla^{-1} x^m &= (q - q^{-1}) \frac{1}{q^{m+1} - q^{-m-1}}\, x^{m+1} \\ &= \frac{1}{[m+1]}\, x^{m+1}\end{aligned} \qquad (5.3)$$

If we apply (5.2) to a field we can expand:

$$\nabla^{-1} f(x) = (q-q^{-1}) \sum_{\nu=0}^{\infty} L^{2\nu} L x f(x) \tag{5.4}$$
$$= -(q-q^{-1}) \sum_{\nu=0}^{\infty} L^{-2\nu} L^{-1} x f(x)$$

These two formulas should be used depending what series converges. That is, for $x^m, m \geq 0$ the first series converges, for $x^m, m < -1$ the second series converges.

A definite integral can be defined only once a representation of the algebra (2.1) is given. We consider a representation where the s of Equation (2.4) is equal to one ($s = 1$). The linear operator x has eigenvalues σq^M. Let us first consider the case $\sigma = +1$. A definite integral would be an integral from q^N to q^M. It should be in agreement with (5.1) for monomials:

$$\int_N^M x^n = \frac{1}{[n+1]} \left(q^{M(n+1)} - q^{N(n+1)} \right) \tag{5.5}$$

For a general field we can extend it by linearity. This definition has Stokes' theorem as a consequence:

$$\int_N^M \nabla x^n = q^{Mn} - q^{Nn} = x^n \Big|_N^M \tag{5.6}$$

The definition (5.5) is not suitable to define an integral over a function in the limit $N \to -\infty, M \to \infty$. To define such an integral we start from Equation (5.4) and apply it to $x^m, m > 0$. We see that the powers of L take even values. Therefore we shall study the integral (5.6) with even and odd M separately, taking immediately the limit $N \to -\infty$.

$$\int_{-\infty}^{2M} x^m = (q-q^{-1}) \sum_{\nu=0}^{\infty} q^{-(2\nu+1)(m+1)} q^{2M(m+1)} \tag{5.7}$$
$$= \frac{1}{[m+1]} q^{2M(m+1)}$$

This agrees with (5.5). Next we rewrite the sum in (5.7)

$$\int_{-\infty}^{2M} x^m = (q-q^{-1}) \sum_{\nu=0}^{\infty} q^{(m+1)(2M-2\nu-1)}$$
$$= (q-q^{-1}) \sum_{\mu=-\infty}^{M} q^{(m+1)(2\mu-1)} \tag{5.8}$$
$$= (q-q^{-1}) \sum_{\mu=-\infty}^{M} \langle 2\mu | L x x^m | 2\mu \rangle$$

where $|2\mu\rangle$ are states of the representation (2.4). We use this formula for polynomials $h(x)$:

$$\int_{-\infty}^{2M} h(x) = (q-q^{-1}) \sum_{\mu=-\infty}^{M} \langle 2\mu | L x h(x) | 2\mu \rangle \tag{5.9}$$

In a similar way we find

$$\int_{-\infty}^{2M+1} h(x) = (q-q^{-1}) \sum_{\mu=-\infty}^{M} \langle 2\mu+1 | L x h(x) | 2\mu+1 \rangle \tag{5.10}$$

For negative powers of $x(x^m, m \leq -2)$ we use the second sum in (5.4) and find for the respective polynomials:

$$\int_{2M}^{\infty} h(x) = (q-q^{-1}) \sum_{\mu=M+1}^{\infty} \langle 2\mu | L x h(x) | 2\mu \rangle \tag{5.11}$$
$$\int_{2M+1}^{\infty} h(x) = (q-q^{-1}) \sum_{\mu=M+1}^{\infty} \langle 2\mu+1 | L x h(x) | 2\mu+1 \rangle$$

It is now obvious how a definite integral for a field should be formulated:

$$\int_{2N}^{2M} h(x) = (q-q^{-1}) \sum_{\mu=N+1}^{M} \langle 2\mu | L x h(x) | 2\mu \rangle \tag{5.12}$$
$$\int_{2N+1}^{2M+1} h(x) = (q-q^{-1}) \sum_{\mu=N+1}^{M} \langle 2\mu+1 | L x h(x) | 2\mu+1 \rangle$$

It is part of an interesting structure that also manifests itself in the q-Fourier transform that odd and even valued lattice points are quite independent. Thus we have two representations of the definite integral over odd or even points. The part of the representation (2.4) with $\sigma = -1$ can be treated completely analogously, but again even and odd are quite independent. For monomials (5.12) is identical with (5.5). This shows that Stokes' theorem holds for even and odd M, N as well:

$$\int_N^M \nabla f(x) = f(x) \Big|_{q^N}^{q^M} \tag{5.13}$$

If for both integrals (5.12) the limit $M \to \infty, N \to -\infty$ exists as well as the corresponding integral over negative eigenvalues of x, then we define the integral as

$$\int h(x) =$$
$$\frac{1}{2}(q-q^{-1}) \sum_{\sigma=+,-} \lim_{\substack{M\to\infty \\ N\to-\infty}} \sigma \Big\{ \sum_{\mu=N}^{M} \langle 2\mu, \sigma | L x h(x) | 2\mu, \sigma \rangle$$
$$+ \langle 2\mu+1, \sigma | L x h(x) | 2\mu+1, \sigma \rangle \Big\} \tag{5.14}$$
$$= \frac{1}{2}(q-q^{-1}) \sum_{\sigma=+,-} \sigma \sum_{\mu=-\infty}^{\infty} \langle \mu, \sigma | L x h(x) | \mu, \sigma \rangle$$

We assume that the rearrangement of the sum is allowed.

It remains to define the integral over x^{-1}. For the definite integral (5.12) this is possible:

$$\int_{2N}^{2M} \frac{1}{x} = (q-q^{-1})(M-N) \tag{5.15}$$

For the integration limits $\bar{z} = q^{2M}, \underline{z} = q^{2N}$ formula (5.15) approaches $\ln \bar{z} - \ln \underline{z}$ for $q \to 1$.

If $h(x) \to 0$ for $x \to \pm\infty$ we conclude from (5.13)

$$\int \nabla h(x) = 0 \tag{5.16}$$

6 The Hilbert space L_q^2

The integral (5.14) can be used to define a scalar product:

$$\begin{aligned}(\chi, \psi) &= \int \chi^* \psi \qquad (6.1)\\ &= \frac{q - q^{-1}}{2} \sum_{\sigma=+1,-1} \sum_{\mu=-\infty}^{\infty} \sigma \langle \mu, \sigma | Lx\chi^*\psi | \mu, \sigma \rangle\end{aligned}$$

The factor Lx in the matrix element allows the sum to converge at $x = 0(q^N, N \to -\infty)$ for fields that do not vanish at $x = 0$. It is the same factor that occurs in the Jackson integral. It should be noted, however, that the integral (6.1) has been obtained in Equation (5.14) from a sum over even and odd values of μ seperately.

For reasonably behaved fields we conclude from Stokes' theorem (5.16) that

$$\int \nabla(\chi^*\psi) = 0 \tag{6.2}$$

and find from (3.10) we find a formula for partial integration:

$$\begin{aligned}&\int (\nabla\chi^*)(L\psi) + \int (L^{-1}\chi^*)(\nabla\psi) = 0 \qquad (6.3)\\ &\int (\nabla\chi^*)(L^{-1}\psi) + \int (L\chi^*)(\nabla\psi) = 0\end{aligned}$$

We use these formulas to find a q-version of Green's theorem. We write:

$$\left.\begin{aligned}\nabla\big((\nabla f)(L^{-1}g)\big) &= (\nabla^2 f)(g) + (L^{-1}\nabla f)(\nabla L^{-1} g)\\ \nabla\big((L^{-1}f)(\nabla g)\big) &= (\nabla L^{-1} f)(L^{-1}\nabla g) + (f)(\nabla^2 g)\end{aligned}\right\} \tag{6.4}$$

We subtract these two equations and obtain Green's theorem:

$$(\nabla^2 f)(g) - (f)(\nabla^2 g) = \nabla\Big((\nabla f)(L^{-1}g) - (L^{-1}f)(\nabla g)\Big) \tag{6.5}$$

As a consequence of (6.2) we find that ∇^2 is an hermitean operator:

$$\int (\nabla^2\chi^*)\psi = \int (\chi^*)(\nabla^2\psi) \tag{6.6}$$

If we integrate (6.5) over a finite volume we find

$$\begin{aligned}&\int_N^M \big\{(\nabla^2 f)(g) - (f)(\nabla^2 g)\big\} = \qquad (6.7)\\ &\big\{(\nabla f)(L^{-1}g) - (L^{-1}f)(\nabla g)\big\}\Big|_N^M\end{aligned}$$

To define ∇^2 we have implicitely used a metric. For a further discussion of this point we refer to Cerchiai *et al.* [10].

7 The $\cos_q$ and $\sin_q$ functions:

The q-deformed cosine and sine functions are defined as follows:

$$\begin{aligned}\sin_q(z) &= \sum_{n=0}^{\infty} (-1)^n q^{-2n(n+1)} \frac{z^{2n+1}}{(q^{-2};q^{-2})_{2n+1}} \qquad (7.1)\\ \cos_q(z) &= \sum_{n=0}^{\infty} (-1)^n q^{-2n(n+1)} \frac{z^{2n}}{(q^{-2};q^{-2})_{2n}}\end{aligned}$$

with:

$$(q^{-2};q^{-2})_n = q^{-\frac{1}{2}n(n+1)}\big(q - q^{-1}\big)^n [n]! \tag{7.2}$$

These functions are well behaved at the points $z = q^{2l}$ and satisfy an orthogonality and completeness relation:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} q^{-2k} \sin_q(q^{-2(k+n)}) \sin_q(q^{-2(k+m)}) &= \frac{q^{2m}}{N_q^2}\delta_{n,m}\\ \sum_{k=-\infty}^{\infty} q^{-2k} \cos_q(q^{-2(k+n)}) \cos_q(q^{-2(k+m)}) &= \frac{q^{2m}}{N_q^2}\delta_{n,m}\end{aligned} \tag{7.3}$$

with

$$N_q = \frac{(q^{-2};q^{-4})_\infty}{(q^{-4};q^{-4})_\infty}.$$

This formula shows that $\cos_q(q^{2l})$ and $\sin_q(q^{2l})$ have to vanish strong enough for $l \to +\infty$ to make the sum in (7.3) converge. This would not be the case for odd powers of q.

Formulas (7.3) tell us that a function f can be expanded in terms of the $\cos_q$ <u>or</u> the $\sin_q$ functions.

$$\begin{aligned}g(q^{-2n}) &= N_q \sum_{k=-\infty}^{\infty} q^{-2k} \cos_q(q^{-2(k+n)}) f(q^{-2k}) \qquad (7.4)\\ g(q^{-2n+1}) &= N_q \sum_{k=-\infty}^{\infty} q^{-2k} \cos_q(q^{-2(k+n)}) f(q^{-2k+1})\end{aligned}$$

The functions g are the expansion coefficients of f and f can be obtained from g with the help of the orthogonality relation (7.3)

$$\begin{aligned}f(q^{-2k}) &= N_q \sum_{n=-\infty}^{\infty} q^{-2n} \cos_q(q^{-2(k+n)}) g(q^{-2n}) \qquad (7.5)\\ f(q^{-2k+1}) &= N_q \sum_{n=-\infty}^{\infty} q^{-2n} \cos_q(q^{-2(k+n)}) g(q^{-2n+1})\end{aligned}$$

We see again that even and odd powers of q play a very independent role. From (7.3) follows that the transformation (7.4) is an isometry:

$$\begin{aligned}&\sum_k q^{-2k} f^*(q^{-2k}) f(q^{-2k}) =\\ &\sum_n q^{-2n} g^*(q^{-2n}) g(q^{-2n}) \qquad (7.6)\end{aligned}$$

$$\sum_k q^{-2k+1} f^*(q^{-2k+1}) f(q^{-2k+1}) = \sum_n q^{-2n+1} g^*(q^{-2n+1}) g(q^{-2n+1})$$

A similar analysis could have been carried out with the $\sin_q$ function.

There is an important relation between $\cos_q$ and $\sin_q$:

$$\frac{1}{z}\big(\cos_q(z) - \cos_q(q^{-2}z)\big) = -q^{-2}\sin_q(q^{-2}z) \quad (7.7)$$
$$\frac{1}{z}\big(\sin_q(z) - \sin_q(q^{-2}z)\big) = \cos_q(z)$$

If we compare this formula with our formula (3.7) for ∇ we find that

$$\nabla\cos_q(x) = -\frac{1}{q}\frac{1}{(q-q^{-1})} L \sin_q(x) \quad (7.8)$$
$$\nabla\sin_q(x) = \frac{q}{(q-q^{-1})} L^{-1}\cos_q(x)$$

Now we want to construct a basis in L^2_q. We split L^2_q into four subspaces with $\sigma = +1$, $\sigma = -1$ and μ even or μ odd.

$$\mathcal{H} = \mathcal{H}^{\mu\,\text{even}}_{\sigma=+1} \oplus \mathcal{H}^{\mu\,\text{odd}}_{\sigma=+1} \oplus \mathcal{H}^{\mu\,\text{even}}_{\sigma=-1} \oplus \mathcal{H}^{\mu\,\text{odd}}_{\sigma=-1} \quad (7.9)$$

On each of these subspaces we define a projector Π^{even}_+, Π^{odd}_+, Π^{even}_- and Π^{odd}_- respectively. We start by defining a basis in $\mathcal{H}^{\mu\,\text{even}}_{\sigma=+1}$. The norm in this subspace is:

$$(\psi,\psi) = \frac{1}{2}(q-q^{-1}) \sum_{\mu-\infty}^{\infty} \langle 2\mu, \sigma=+1 | Lx\psi^*\psi | 2\mu, \sigma=+1\rangle \quad (7.10)$$

The functions:

$$\mathcal{C}^{(+)}_{\substack{2n+1\\ n\in\mathbb{Z}}}(x) = \tilde{N}_q \cos_q(xq^{2n+1})\Pi^{\text{even}}_+, \quad (7.11)$$
$$\tilde{N}_q = N_q\Big(\frac{2q}{q-q^{-1}}\Big)^{\frac{1}{2}} q^n$$

form an orthonormal basis in $\mathcal{H}^{\mu\,\text{even}}_{\sigma=+1}$. This follows from (7.4) and (7.5) and the definition of the scalar product (7.10). Similarly we define

$$\mathcal{C}^{(+)}_{2n}(x) = \tilde{N}_q \cos_q(xq^{2n})\Pi^{\text{odd}}_+ \quad (7.12)$$

in the subspace $\mathcal{H}^{\mu\,\text{odd}}_{\sigma=+1}$ with the norm

$$(\psi,\psi) = \frac{1}{2}(q-q^{-1}) \sum_{\mu-\infty}^{\infty} \langle 2\mu+1, \sigma=+1 | Lx\psi^*\psi | 2\mu+1, \sigma=+1\rangle \quad (7.13)$$

and

$$\mathcal{C}^{(-)}_{2n+1}(x) = \tilde{N}_q \cos_q(xq^{2n+1})\Pi^{\text{even}}_- \quad (7.14)$$
$$\mathcal{C}^{(-)}_{2n}(x) = \tilde{N}_q \cos_q(xq^{2n})\Pi^{\text{odd}}_-$$

In an analogous way we could have used $\sin_q$ in any of the subspaces to define $\mathcal{S}^{(+)}_{2n+1}(x)$, $\mathcal{S}^{(+)}_{2n}(x)$, $\mathcal{S}^{(-)}_{2n+1}(x)$ and $\mathcal{S}^{(-)}_{2n}(x)$ any combination of the individual basis can be used to define a basis in all of L^2_q. We note, however, that the operator ∇ is not defined on any of the elements of this basis. It maps elements of the basis into functions that are not in L^2_q. In contrast ∇L or ∇L^{-1} is defined on this basis. These are exactly the operators that enter in Green's theorem (6.5) if we write it in the form:

$$(\nabla^2 f)(g) - (f)(\nabla^2 g) = \nabla L^{-1}\big\{(L\nabla f)(g) - (f)(L\nabla g)\big\} \quad (7.15)$$

The elements of our basis are eigenfunctions of the operator ∇^2. To see this we generalize (7.8)

$$\nabla_x\cos_q(xy) = \frac{1}{q-q^{-1}} x^{-1}\big(\cos_q(qxy) - \cos_q(q^{-1}xy)\big) = -\frac{1}{q-q^{-1}} q^{-1} y L \sin_q(xy) \quad (7.16)$$
$$\nabla_x\sin_q(xy) = \frac{1}{q-q^{-1}} q y L^{-1}\cos_q(xy)$$

and we find

$$\nabla^2\cos_q(xy) = -\frac{1}{q}\big(q-q^{-1}\big)^{-2} y^2 \cos_q(xy) \quad (7.17)$$
$$\nabla^2\sin_q(xy) = -q\big(q-q^{-1}\big)^{-2} y^2 \sin_q(xy)$$

We see that the basis represents eigenfunctions with the following eigenvalues:

$$\begin{aligned} \mathcal{C}^{+,-}_{2n+1} &: -\big(q-q^{-1}\big)^{-2} q^{4n+1} \\ \mathcal{C}^{+,-}_{2n} &: -\big(q-q^{-1}\big)^{-2} q^{4n-1} \\ \mathcal{S}^{+,-}_{2n+1} &: -\big(q-q^{-1}\big)^{-2} q^{4n+3} \\ \mathcal{S}^{+,-}_{2n} &: -\big(q-q^{-1}\big)^{-2} q^{4n+1} \end{aligned} \quad (7.18)$$

We also see that the set of eigenfunctions is overcomplete. We conclude that ∇^2 is hermitian but not self-adjoint. On any of the bases defined before a self-adjoint extension of the operator ∇^2 is defined. Note that the set of eigenvalues does depend on the respective extension.

8 Schrödinger equation

It is natural to define the Schrödinger equation as follows:

$$i\frac{\partial}{\partial t}\psi = \big(-\frac{1}{2m}\nabla^2 + V\big)\psi \quad (8.1)$$

It has a probability interpretation. To show this we calculate:

$$\frac{d}{dt}\psi^*\psi = -\frac{1}{2mi}\big\{\psi^*(\nabla^2\psi) - (\nabla^2\psi^*)\psi\big\} \quad (8.2)$$

We can use (6.5):

$$\frac{d}{dt}\psi^*\psi = -\frac{1}{2mi}\nabla\big\{(L^{-1}\psi^*)(\nabla\psi) - (\nabla\psi^*)(L^{-1}\psi)\big\} = -\frac{1}{2mi}\nabla L^{-1}\big\{\psi^*(L\nabla\psi) - (L\nabla\psi^*)\psi\big\} \quad (8.3)$$

This is the continuity equation for the density

$$\rho = \psi^*\psi \tag{8.4}$$
$$j = \frac{1}{2mi} L^{-1}\{\psi^*(L\nabla\psi) - (L\nabla\psi^*)\psi\}$$
$$\frac{\partial}{\partial t}\rho + \nabla j = 0$$

If we integrate this equation over a finite volume we obtain from (5.13):

$$\frac{d}{dt}\int_{2N}^{2M} \psi^*\psi = -\frac{1}{2mi}\left\{L^{-1}\big(\psi^*(L\nabla\psi) - (L\nabla\psi^*)\psi\big)\right\}\Big|_{2N}^{2M} \tag{8.5}$$

Note that the "velocity" operator is proportional to $L\nabla$. If we consider the free Schrödinger equation we know the solutions of the time-independent equation:

$$-\frac{1}{2m}\nabla^2\psi = E\psi \tag{8.6}$$

These are the functions $\mathcal{S}$ and $\mathcal{C}$ defined in eqn. (7.11), (7.12) and (7.14). The eigenvalues can be found in (7.18).

As an example let us consider the eigenfunction $\mathcal{C}^{(+)}_{2n+1}$. The energy eigenvalue is

$$E^{(+)}_{2n+1} = \frac{(2m)^{-1}}{\left(q - q^{-1}\right)^2} q^{4n+1} \tag{8.7}$$

The normalized eigenfunction is

$$\psi^{(+)}_{2n+1} = \tilde{N}_q \cos_q(xq^{2n+1}) \Pi_+^{\text{even}} \tag{8.8}$$

For the probability of finding the "particle" in the volume between q^{2N} and q^{2M} we obtain

$$\begin{aligned}
&\int_{2N}^{2M} \tilde{N}_q^2 \cos_q^2(xq^{2n+1}) \\
&= 2N_q^2 q^{2n+1} \sum_{\mu=N+1}^{M} \langle 2\mu | Lx \cos_q^2(xq^{2n+1}) | 2\mu\rangle \\
&= 2N_q^2 q^{2n} \sum_{\mu=N+1}^{M} q^{2\mu} \cos_q^2(q^{2\mu+2n})
\end{aligned} \tag{8.9}$$

We see that the $\cos_q$ function is well defined for even powers of q. For the time derivative we evaluate the right hand side of Equation (8.2). A typical term is:

$$L^{-1}\big(\psi^*(L\nabla\psi) - (L\nabla\psi^*)\psi\big)\Big|_{\text{at } x=q^{2M}} \tag{8.10}$$

We use (7.8) and again find that the $\cos_q$ and $\sin_q$ functions are well defined and that the flux is zero. The eigenfunctions (8.8) are stationary.

9 Covariant derivatives

We assume that the field $\psi(x)$ transforms under a gauge transformation as follows:

$$\psi'(x) = e^{i\alpha(x)}\psi(x) \tag{9.1}$$

This is not necessarily an abelian gauge group. We would like to define a derivative $\mathcal{D}\psi$, such that

$$(\mathcal{D}\psi)' = e^{i\alpha}\mathcal{D}\psi \tag{9.2}$$

For an ordinary derivative we know from the generalized Leibniz rule (3.10) that:

$$\nabla\psi' = \left(\nabla e^{i\alpha}\right)(L\psi) + \left(L^{-1}e^{i\alpha}\right)\nabla\psi \tag{9.3}$$

For a covariant derivative we make the Ansatz:

$$\mathcal{D}\psi = E(\nabla + \phi)\psi \tag{9.4}$$

The field E plays the role of an Einbein and ϕ the role of a connection. We now determine the transformation law of E and ϕ from (9.1) and (9.2):

$$E'(\nabla + \phi')e^{i\alpha}\psi = e^{i\alpha}E(\nabla + \phi)\psi \tag{9.5}$$

We find

$$E' = e^{i\alpha}E\left(L^{-1}e^{-i\alpha}\right) \tag{9.6}$$

and

$$\phi' = \left(L^{-1}e^{i\alpha}\right)\phi\left(e^{-i\alpha}\right) - \left(\nabla e^{i\alpha}\right)Le^{-i\alpha} \tag{9.7}$$

Note that the last term is without bracket, the "operator" L acts on the field that multiplies ϕ' as well. Another way of saying this is that ϕ is L-valued:

$$\phi = \varphi L \tag{9.8}$$

For φ we find from (9.7):

$$\varphi' = \left(L^{-1}e^{i\alpha}\right)\varphi\left(Le^{-i\alpha}\right) - \left(\nabla e^{i\alpha}\right)\left(Le^{-i\alpha}\right) \tag{9.9}$$

We would like to show now that φ can be considered to be a function of E. To do this we recall formula (3.7) and define a covariant operator $\mathcal{L}$ by

$$(\mathcal{L}\psi)' = e^{i\alpha}\mathcal{L}\psi \tag{9.10}$$

From the comultiplication law of L follows that

$$L\psi' = \left(Le^{i\alpha}\right)(L\psi) \tag{9.11}$$

We make the Ansatz:

$$\mathcal{L}\psi = \tilde{E}L\psi \tag{9.12}$$

and find the transformation law of $\tilde{E}$ from (9.1) and (9.10):

$$\tilde{E}' = e^{i\alpha}\tilde{E}\left(Le^{-i\alpha}\right) \tag{9.13}$$

Similarly we define:

$$\tilde{\mathcal{L}}\psi = EL^{-1}\psi \tag{9.14}$$

and find

$$E' = e^{i\alpha} E\big(L^{-1} e^{-i\alpha}\big) \tag{9.15}$$

This transformation law agrees with (9.6) and this justifies our choice of E in the definition (9.14).

The inverse Einbein E^{-1} transforms as follows:

$$E^{-1\prime} = \big(L^{-1} e^{i\alpha}\big) E^{-1} e^{-i\alpha} \tag{9.16}$$

We see that it is a consistent assumption to postulate

$$\tilde{E} = \big(LE^{-1}\big) \tag{9.17}$$

With this assumption we find that

$$\tilde{\mathcal{L}} = \mathcal{L}^{-1} \tag{9.18}$$

We generalize formula (3.7):

$$\mathcal{D} = \frac{1}{\lambda} x^{-1} \big(\tilde{\mathcal{L}} - \mathcal{L}\big) \tag{9.19}$$

and we know from (9.13) and (9.16) that this definition of $\mathcal{D}$ has the right transformation property. We now rewrite (9.19) to compare it with (9.4)

$$\begin{aligned} \mathcal{D} &= \frac{1}{\lambda} x^{-1} \big\{E(L^{-1} - L) + (E - \tilde{E})L\big\} \\ &= E\nabla + \frac{1}{\lambda} x^{-1} (E - \tilde{E}) L \end{aligned} \tag{9.20}$$

We again see that the connection is L valued and find:

$$\varphi = \frac{1}{\lambda} x^{-1} (1 - E^{-1}\tilde{E}) \tag{9.21}$$

It takes a small calculation to verify that φ has the transformation property (9.9) as a consequence of (9.13) and (9.15). It is interesting to note that the covariant derivative of the Einbein E vanishes if we choose (9.21) for the connection. We start from a field H that transforms like E:

$$H' = e^{i\alpha} H\big(L^{-1} e^{-i\alpha}\big) \tag{9.22}$$

Let us first compute $\mathcal{L}$ and $\tilde{\mathcal{L}}$ applied to H:

$$\begin{aligned} \mathcal{L}H &= \tilde{E}(LH)\big(L^{-1}\tilde{E}^{-1}\big) \\ \tilde{\mathcal{L}}H &= E\big(L^{-1}H\big)\big(L^{-1}E^{-1}\big) \end{aligned} \tag{9.23}$$

We now identify $\tilde{E}$ with $\big(LE^{-1}\big)$ as in (9.17) and find

$$\mathcal{L}H = \big(LE^{-1}\big)(LH)(E) \tag{9.24}$$

Now we identify H with E and obtain

$$\mathcal{L}E = E, \qquad \tilde{\mathcal{L}}E = E \tag{9.25}$$

Combining this with a covariant derivative as in (9.19) yields

$$\mathcal{D}E = 0 \tag{9.26}$$

10 Leibniz rule for covariant derivatives

We first multiply the field ψ by a scalar field f:

$$f' = f, \quad \mathcal{D}f = \nabla f, \quad \mathcal{L}f = Lf, \quad \tilde{\mathcal{L}}f = L^{-1}f \tag{10.1}$$

and we compute

$$\mathcal{L}f\psi = \tilde{E}Lf\psi = \tilde{E}(Lf)(L\psi) = \mathcal{L}f\mathcal{L}\psi \tag{10.2}$$

Thus we find for $\mathcal{L}$ the same comultiplication rule as for L. Next we apply this comultiplication rule to the scalar $\psi^*\psi$:

$$\begin{aligned} \mathcal{L}\psi^*\psi &= L(\psi^*\psi) = (L\psi^*)(L\psi) \\ &= (L\psi^*)\tilde{E}^{-1}(\tilde{E}L\psi) = \mathcal{L}\psi^*\mathcal{L}\psi \end{aligned} \tag{10.3}$$

and we conclude that

$$\mathcal{L}\psi^* = (L\psi^*)\tilde{E}^{-1} = (L\psi^*)(LE) \tag{10.4}$$

We have here used (9.17).

If we now apply the comultiplication rule to the product of two arbitrary representations we find:

$$\begin{aligned} \mathcal{L}\psi\chi &= \mathcal{L}\psi\mathcal{L}\chi \\ &= (\tilde{E}L)\psi(\tilde{E}L\chi) \end{aligned} \tag{10.5}$$

It becomes more transparent if we use indices:

$$\begin{aligned} \mathcal{L}\psi_\mu\chi_\alpha &= \tilde{E}_\mu{}^\rho \tilde{E}_\alpha{}^\beta (L\psi)_\rho (L\psi)_\beta \\ &= \tilde{E}_{\mu\alpha}{}^{\rho\beta} L\psi_\rho \chi_\beta \end{aligned} \tag{10.6}$$

Thus the Einbein matrix for the product representation is the product of the Einbein matrices of the respective representations. Therefore we can generate the Einbein for any representation from the Einbein of the fundamental representation. This immediately generalizes to $\tilde{\mathcal{L}}$ and $\mathcal{D}$. We find the comultiplication rule

$$\tilde{\mathcal{L}}\psi\chi = \tilde{\mathcal{L}}\psi\tilde{\mathcal{L}}\chi \tag{10.7}$$

and the Leibniz rule

$$\mathcal{D}\psi\chi = (\mathcal{D}\psi)\mathcal{L}\chi + (\tilde{\mathcal{L}}\psi)\mathcal{D}\chi \tag{10.8}$$

It should be noted that the transformation parameter α in (9.1) is proportional to a coupling constant g. Thus we should expand in the coupling constant:

$$E_\alpha{}^\beta = \delta_\alpha{}^\beta + g h_\alpha{}^\beta(x) \tag{10.9}$$

11 Curvature

To be able to speak about curvature we must first define a covariant time derivative:

$$\mathcal{D}_t\psi = (\partial_t + \omega)\psi \tag{11.1}$$

The connection ω transforms as follows:

$$\omega' = e^{i\alpha}\omega e^{-i\alpha} + e^{i\alpha}\partial_t e^{-i\alpha} \tag{11.2}$$

Now we can compute the commutator of the two covariant derivatives:

$$
\begin{aligned}
&(\mathcal{D}_t\mathcal{D} - \mathcal{D}\mathcal{D}_t)\psi = \\
&\big\{(\partial_t + \omega)E(\nabla + \phi) - E(\nabla + \phi)(\partial_t + \omega)\big\}\psi \\
&= \big\{(\partial_t E)E^{-1} - E(L^{-1}\omega)E^{-1} + \omega\big\}\mathcal{D}\psi \\
&+\big\{\partial_t E\phi - E(\nabla\omega)L - E\phi\omega - (\partial_t E)\phi + E(L^{-1}\omega)\phi\big\}\psi
\end{aligned} \tag{11.3}
$$

This allows us to define two tensors:

$$
\begin{aligned}
T &= (\partial_t E)E^{-1} - (E)(L^{-1}\omega)E^{-1} + \omega \\
F &= \big\{\partial_t\varphi - \nabla\omega + (L^{-1}\omega)\varphi - \varphi(L\omega)\big\}
\end{aligned} \tag{11.4}
$$

and to write (11.3) in the form

$$
(\mathcal{D}_t\mathcal{D} - \mathcal{D}\mathcal{D}_t)\psi = T\mathcal{D}\psi + EFL\psi \tag{11.5}
$$

Starting from the transformation laws of E, ω and φ, these are the eqn. (9.6), (9.9) and (11.2). We can verify by a lengthy calculation the transformation laws of T and F as they follow from the definition (11.5):

$$
\begin{aligned}
T' &= e^{i\alpha}Te^{-i\alpha} \\
F' &= \big(L^{-1}e^{i\alpha}\big)F\big(Le^{-i\alpha}\big)
\end{aligned} \tag{11.6}
$$

The tensor T has already the right transformation law. It follows from (9.6) that the quantity

$$
\mathcal{F} = EF(LE) \tag{11.7}
$$

transforms as a tensor as well:

$$
\mathcal{F}' = e^{i\alpha}\mathcal{F}e^{-i\alpha} \tag{11.8}
$$

The curvature T can also be derived from the following commutation:

$$
(\mathcal{L}\mathcal{D}_t - \mathcal{D}_t\mathcal{L})\psi = LE^{-1}T\psi \tag{11.9}
$$

12 Euler-Lagrange equation and Noether theorem

The definition of the integral (5.12) allows the formulation of a variational problem. An action can be defined as the integral over a Lagrangian. The Lagrangian itself is a function of the fields and their derivatives and we demand that the action be extremal under the variation of the fields with fixed boundary values. Let us first examine the Schrödinger equation (8.1). We define the action

$$
W = \int_{t_1}^{t_2} dt \int_{2N}^{2M} \psi^*\big(i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\nabla^2\psi - V\psi\big) \tag{12.1}
$$

Variation of ψ^* leads to the Schrödinger equation (8.1). We know that ∇^2 is hermitian. Thus a variation of ψ leads to the conjugate Schrödinger equation.

We would prefer to formulate the Lagrangian in terms of the fields and their first derivatives. To write (12.1) in such a form we note that the Leibniz rule (3.10) implies that

$$
\int(\nabla L\psi)\chi = -\int\psi(\nabla L^{-1}\chi) \tag{12.2}
$$

The adjoint operator of ∇L^{-1} is $-\nabla L$, given by

$$
\big(\nabla L^{-1}\big)^+ = -\nabla L \tag{12.3}
$$

The Laplacian ∇^2 can be written as $\nabla LL^{-1}\nabla$ and we obtain in this way:

$$
\begin{aligned}
\int\psi^*\nabla^2\psi &= \int\psi^*\nabla LL^{-1}\nabla\psi \\
&= -\frac{1}{q}\int\big(\nabla L^{-1}\psi^*\big)\big(\nabla L^{-1}\psi\big)
\end{aligned} \tag{12.4}
$$

An equivalent action to (12.1) is:

$$
\int_{t_1}^{t_2} dt \int_{2N}^{2M} \big\{i\psi^*\frac{\partial}{\partial t}\psi - \frac{1}{2mq}\big(\nabla L^{-1}\psi^*\big)\big(\nabla L^{-1}\psi\big) - V\psi^*\psi\big\} \tag{12.5}
$$

We have a Lagrangian that depends on ψ, $\dot\psi$ and $\nabla L^{-1}\psi$ as well as on the conjugate expresions.

Let us now assume that we have a Lagrangian which depends on the fields ψ, $\dot\psi$ and $\nabla L^{-1}\psi$. The ψ^* is considered to be an independent field. The variation of the action due to the variation of the fields:

$$
\psi' = \psi + \delta\psi, \qquad \delta\psi(q^{2M}, t_2) = \delta\psi(q^{2N}, t_1) = 0 \tag{12.6}
$$

can be written as follows:

$$
\begin{aligned}
\delta W &= \int_{t_1}^{t_2} dt \int_{2N}^{2M} \Big\{\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\dot\psi}\delta\dot\psi + \frac{\partial\mathcal{L}}{\partial\,\nabla L^{-1}\psi}\delta\nabla L^{-1}\psi\Big\} \\
&= \int_{t_1}^{t_2} dt \int_{2N}^{2M} \Big\{\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\dot\psi}\frac{\partial}{\partial t}\delta\psi + \frac{\partial\mathcal{L}}{\partial\,\nabla L^{-1}\psi}\nabla L^{-1}\delta\psi\Big\} \\
&= \int_{t_1}^{t_2} dt \int_{2N}^{2M} \Big\{\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot\psi} - \nabla L\frac{\partial\mathcal{L}}{\partial\,\nabla L^{-1}\psi}\Big\}\delta\psi
\end{aligned} \tag{12.7}
$$

The last step involves the fact that the variations of the fields vanish at the boundary. We obtain thus the Euler-Lagrange equation:

$$
\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot\psi} + \nabla L\frac{\partial\mathcal{L}}{\partial\,\nabla L^{-1}\psi} \tag{12.8}
$$

It is easy to verify that eqn. (12.8) yields the Schrödinger equation (8.1) for the action (12.5).

To formulate the Noether theorem we study variations of the field ψ at the same spacetime point

$$
\psi'(x) = \psi(x) + \Delta\psi(x) \tag{12.9}
$$

which leave the action W invariant

$$
\delta W = \int dt\int\big(\mathcal{L}' - \mathcal{L}\big) = 0 \tag{12.10}
$$

We expand $\mathcal{L}'$,

$$\mathcal{L}' = \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\psi}\Delta\psi + \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\Delta\dot{\psi} + \frac{\partial\mathcal{L}}{\partial\nabla L^{-1}\psi}\Delta(\nabla L^{-1}\psi) \quad (12.11)$$

and we find that

$$\Delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\Delta\psi + \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\Delta\dot{\psi} + \frac{\partial\mathcal{L}}{\partial\nabla L^{-1}\psi}(\nabla L^{-1}\Delta\psi) = 0 \quad (12.12)$$

For the first term we insert the Euler-Lagrange equation (12.8) and we obtain

$$\frac{\partial}{\partial t}\Big\{\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\Delta\psi\Big\} + \nabla\Big\{\Big(L\frac{\partial\mathcal{L}}{\partial\nabla L^{-1}\psi}\Big)(L^{-1}\Delta\psi)\Big\} = 0 \quad (12.13)$$

This is Noether's theorem. If we apply it to the Lagrangian (12.5) and the phase transformation we obtain

$$\psi' = e^{i\alpha}\psi, \quad \Delta\psi = i\alpha\psi, \quad \Delta\psi^* = -i\alpha\psi^* \quad (12.14)$$

For the "charge" density we find from (12.13)

$$-\alpha\rho = \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\Delta\psi + \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\Delta\psi^* = -\alpha\psi^*\psi \quad (12.15)$$

and for the current

$$\begin{aligned}
&\Big(L\frac{\partial\mathcal{L}}{\partial\nabla L^{-1}\psi}\Big)(L^{-1}\Delta\psi) + \Big(L\frac{\partial\mathcal{L}}{\partial\nabla L^{-1}\psi^*}\Big)(L^{-1}\Delta\psi^*) = \\
&= -\frac{i\alpha}{2mq}\big\{(L\nabla L^{-1}\psi^*)(L^{-1}\psi) - (L\nabla L^{-1}\psi)(L^{-1}\psi^*)\big\} \\
&= +\alpha\frac{1}{2im}\big\{(\nabla\psi^*)(L^{-1}\psi) - (\nabla\psi)(L^{-1}\psi^*)\big\} \quad (12.16) \\
&= -\alpha\frac{1}{2im}L^{-1}\big\{\psi^*(L\nabla\psi) - (L\nabla\psi^*)\psi\big\} \\
&= -\alpha j
\end{aligned}$$

This agrees with our definition of j in eqn. (8.4). There we have verified explicitely the continuity equation (8.3).

13 The q-Harmonic oscillator

In analogy to [6] we define a q-deformation of the Harmonic Oscillator with the help of a creation and annihilation operator:

$$\begin{aligned} a &= \alpha L^{-2} - i\beta\nabla L^{-1} \\ a^+ &= \bar{\alpha}q^{-2}L^2 - i\bar{\beta}\nabla L \end{aligned} \quad (13.1)$$

They satisfy the q-commutation relation:

$$aa^+ = q^{-2}a^+a + q^{-2}(1-q^{-2})|\alpha|^2 \quad (13.2)$$

To normalise this equation we set:

$$|\alpha| = \frac{q}{\sqrt{1-q^{-2}}} \quad (13.3)$$

Here α and β are only up to q-factors equal to the respective constants in [6].

A general expression for the operators in (13.1) can be found:

$$\begin{aligned} a &= \alpha L^{-2m} - i\beta L^{-m-1}\nabla L \\ a^+ &= \bar{\alpha}q^{-2m}L^{2m} - iq^{-m-1}\bar{\beta}\nabla L^{-1}L^{m+1} \end{aligned} \quad (13.4)$$

with the commutation relation:

$$aa^+ - q^{-2m}a^+a = q^{-2m}(1-q^{-2m})|\alpha|^2 \quad (13.5)$$

The Hamiltonian for the q-deformed oscillator based on (13.1), (13.2) has the following form:

$$\begin{aligned} H &= a^+a \\ &= |\alpha|^2q^{-2} - i\bar{\alpha}\beta\nabla L - i\alpha\bar{\beta}\nabla L^{-1} - q|\beta|^2\nabla^2 \end{aligned} \quad (13.6)$$

and the Schrödinger equation

$$i\partial_t\psi = \left(-q|\beta|^2\nabla^2 - i\bar{\alpha}\beta\nabla L - i\alpha\bar{\beta}\nabla L^{-1} + q^{-2}|\alpha|^2\right)\psi \quad (13.7)$$

The Lagrange function to this equation is:

$$\begin{aligned} \mathcal{L} = {} & i\psi^*\partial_t\psi - |\beta|^2\left(\nabla L^{-1}\psi^*\right)\left(\nabla L^{-1}\psi\right) \\ & -i\bar{\alpha}\beta\left(\nabla L^{-1}\psi^*\right)\psi + i\alpha\bar{\beta}\psi^*\left(\nabla L^{-1}\psi\right) \\ & -|\alpha|^2q^{-2}\psi^*\psi \end{aligned} \quad (13.8)$$

Now we have a look at the spectrum. We first consider the ground state. The Hamiltonian of the q-deformed Harmonic Oscillator is a positive operator. So we define a ground state by:

$$a|0\rangle = 0 \quad (13.9)$$

We have a q-difference equation:

$$\alpha L^{-2}\psi_0(x) = i\beta\nabla L^{-1}\psi_0(x) \quad (13.10)$$

When we consider an Ansatz with a q-deformed exponential function:

$$e_{q^{-2}}(x) \equiv \sum_{k=0}^{\infty}\frac{x^k}{(q^{-2};q^{-2})_k} \quad (13.11)$$

we find for $\psi_0(x)$:

$$\psi_0(x) = e_{q^{-2}}\left(-i\frac{\lambda\alpha}{q^2\beta}x\right) \quad (13.12)$$

This function can be seen to be the q-Fourier transform of the gaussian function:

$$f(q^l) = q^{-\frac{1}{2}(l^2+l)}c_0 \quad (13.13)$$

We consider the q-Fourier transformation of (7.4) and consider the Ansatz:

$$g(\tau q^{2\nu}) = \frac{N_q}{\sqrt{2}}\sum_{l=-\infty}^{\infty}q^{\nu+l}\Big(f(q^{2l})\cos_q 2(\nu+l) + i\tau f(q^{2l+1})\sin_q 2(\nu+l)\Big) \quad (13.14)$$

Inserting the definitions of $\cos_q 2(\nu+l)$, $\sin_q 2(\nu+l)$ and noting that the sum over l can be cast into a constant by a Gauss-summation:

$$\sum_{l=-\infty}^{\infty} q^{-2(l-n)^2} c_0 \equiv \tilde{c_0} \tag{13.15}$$

which takes the value (by Jacobi's Triple Product Identity [3]):

$$\tilde{c_0} = c_0(q^{-4}, -q^{-2}, -q^{-2}; q^{-4})_\infty \tag{13.16}$$

we find that

$$\begin{aligned} g(\tau q^{2\nu}) &= \frac{N_q}{\sqrt{2}} \tilde{c_0} q^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{-2n}}{(q^{-2};q^{-2})_{2n}} q^{4\nu n} \\ &\quad + i\tau \sum_{n=0}^{\infty} (-1)^n \frac{q^{-(2n+1)}}{(q^{-2};q^{-2})_{2n+1}} q^{2\nu(2n+1)} \\ &= \frac{N_q}{\sqrt{2}} \tilde{c_0} q^\nu e_{q^{-2}}(i\tau q^{-1} q^{2\nu}) \end{aligned} \tag{13.17}$$

The two sums are precisely the real and imaginary part of a q-deformed exponential function. The function $f(q^l)$ is the ground state function from [6] (with $\frac{\alpha}{\beta}=1$) which was calculated in the momentum basis. So we explicitly calculated the q-Fourier transformation of the ground state in momentum space to the ground state in configuration space which is a q-deformed exponential function. For the odd components we consider the Ansatz:

$$\begin{aligned} g(\tau q^{2\nu+1}) = \frac{N_q}{\sqrt{2}} \sum_{l=-\infty}^{\infty} q^{\nu+l} \big(f(q^{2l+1}) q \cos_q 2(\nu+l+1) \\ + i\tau f(q^{2l}) \sin_q 2(\nu+l) \big) \end{aligned} \tag{13.18}$$

and find by an analogous calculation:

$$g(\tau q^{2\nu+1}) = \frac{N_q}{\sqrt{2}} c_0' q^\nu e_{q^{-2}}(i\tau q^{-1} q^{2\nu+1}) \tag{13.19}$$

The constant c_0' is different from $\tilde{c_0}$:

$$c_0' = c_0(q^{-4}, -q^{-4}, -1; q^{-4})_\infty \tag{13.20}$$

The functions (13.17) and (13.19) are the same as (13.12) evaluated in the representation of [9] on even and odd lattice points respectively:

$$X|\nu,\tau\rangle = -\tau \frac{q^\nu q^{-\frac{1}{2}}}{\lambda} |\nu,\tau\rangle \tag{13.21}$$

Note that in comparison to the Notation in [6] the constant $\frac{\alpha}{\beta}$ here takes the value $q^{\frac{3}{2}}$.

Next we consider the excited states of the q-oscillator. We follow the arguments of [6] and find that the q-deformed Hermite Polynomials appear in the same way. First we note that:

$$a^+|0\rangle = i \frac{1}{q^{\frac{1}{2}}\beta} X|0\rangle \tag{13.22}$$

which is easily verified with the help of (13.1) and (13.9). To show that the n-particle state can analogously be expressed by a polynomial in X:

$$\left(a^+\right)^n |0\rangle = \left(\frac{1}{\sqrt{2}}\right)^n H_n^{(q)} \left(\frac{iX}{\sqrt{2}\beta}\right) |0\rangle \tag{13.23}$$

We calculate the commutation relation between a^+ and ξ, where ξ is dimensionless:

$$\xi \equiv \frac{i}{\sqrt{2}\beta} X \tag{13.24}$$

and find:

$$a^+\xi = q^{-2}\xi a^+ - \frac{q^{-\frac{3}{2}}}{\sqrt{2}} \tag{13.25}$$

This leads to the same recursion relation as in [6] for the polynomials $H_n^{(q)}(\xi)$, because the recursion relation can be proven by induction over n with the help of (13.25):

$$H_{n+1}^{(q)}(\xi) - q^{-\frac{1}{2}} q^{-2n} 2\xi H_n^{(q)}(\xi) + 2q^{-n-1}[n] H_{n-1}^{(q)}(\xi) = 0 \tag{13.26}$$

We have used here the symmetric q-number

$$[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}$$

Finally we want to consider another example of the q-Fourier transform of [5]. We calculate the q-Fourier transform of the step function which will be useful for further applications. We define the step function

$$\Theta(q^{2n} - q^{2M}) \equiv \begin{cases} 1 & ,n \le M \\ 0 & ,n > M \end{cases} \tag{13.27}$$

Now we calculate the q-Fourier transform of this function:

$$\begin{aligned} \widetilde{\Theta}(q^{2k} - q^{2M}) &= N_q \sum_{n=-\infty}^{\infty} q^{2n} \cos_q(q^{2(k+n)} \Theta(q^{2n} - q^{2M}) \\ &= N_q \sum_{n=-\infty}^{M} q^{2n} \cos_q(q^{2(k+n)}) \end{aligned} \tag{13.28}$$

The sum can be calculated with the help of the integral (5.10) and the fact that $\cos_q(z)$ can be expressed as the q-derivative of $\sin_q(z)$ (7.16). We find using Stokes theorem

$$\widetilde{\Theta}(q^{2k} - q^{2M}) = N_q q^{-2k} \sin_q(q^{2(k+M)}) \tag{13.29}$$

This result can also be obtained by applying the q-difference relation (7.7) to the sum in (13.28).

To verify our calculation we try to perform the q-Fourier transformation in the other direction, too:

$$\begin{aligned} g(q^{2n}) &= N_q \sum_{k=-\infty}^{\infty} q^{2k} \cos_q(q^{2(k+n)} \widetilde{\Theta}(q^{2k} - q^{2M}) \\ &= N_q^2 \sum_{k=-\infty}^{\infty} \cos_q(q^{2(k+n)} \sin_q(q^{2(k+M)}) \end{aligned} \tag{13.30}$$

The sum on the right hand side can be written in terms of q-Bessel functions, for the notation see [5]:

$$g(q^{2n}) = q^{M+n} \sum_{k=-\infty}^{\infty} q^{-2k} J_{-\frac{1}{2}}(q^{2(n-k)}) J_{\frac{1}{2}}(q^{2(M-k)}) \tag{13.31}$$

To calculate this sum we use the summation formula for q-Bessel functions from [5] and find

$$g(q^{2n}) = q^{M+n}$$
$$\cdot \begin{cases} q^{M-3n} \frac{(1,q^{-6};q^{-4})_\infty}{(q^{-2},q^{-4};q^{-4})_\infty} {}_2\Phi_1(q^{-4}, q^{-2}; q^{-6}; q^{-4}, q^{-4(n-M)}) \\ q^{-(M+n)} \frac{(q^{-4},q^{-2};q^{-4})_\infty}{(q^{-2},q^{-4};q_\infty^{-4})} {}_2\Phi_1(1, q^{-2}; q^{-2}; q^{-4}, q^{-4(M-n+1)}) \end{cases}$$

Because of the finite radius of convergence of the hypergeometric series which is $r = 1$ in this case [3], we get different conditions for the variable n. Noting that

$$(1; q^{-4})_\infty = 0 \tag{13.32}$$

We find

$$g(q^{2n}) = \begin{cases} 0 & , n > M \\ 1 & , n \leq M \end{cases} \tag{13.33}$$

This is exactly the step function we defined in (13.27).

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